Modelling of Navier-Stokes Equations

Challenges and Improvements

Group 9

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Lehrstuhl für Hydromechanik

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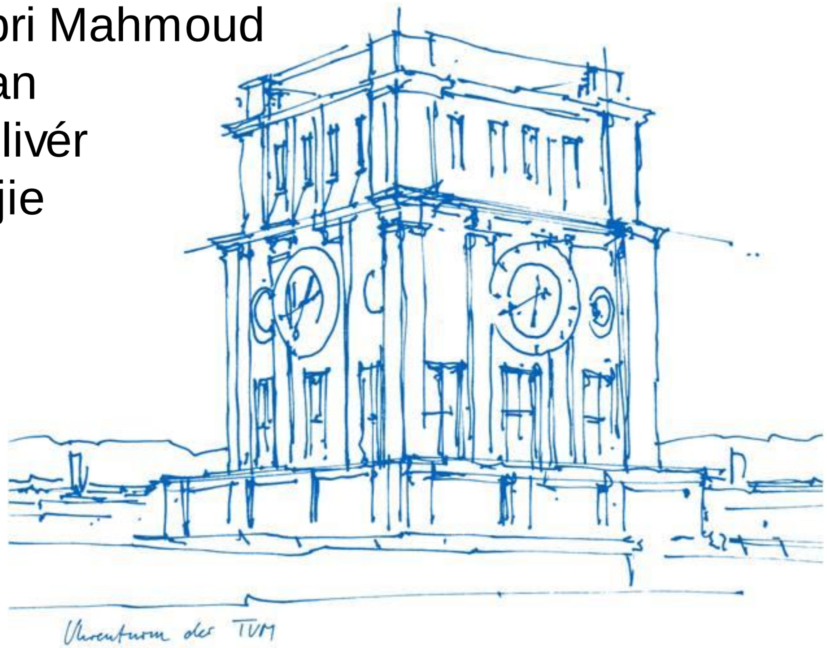

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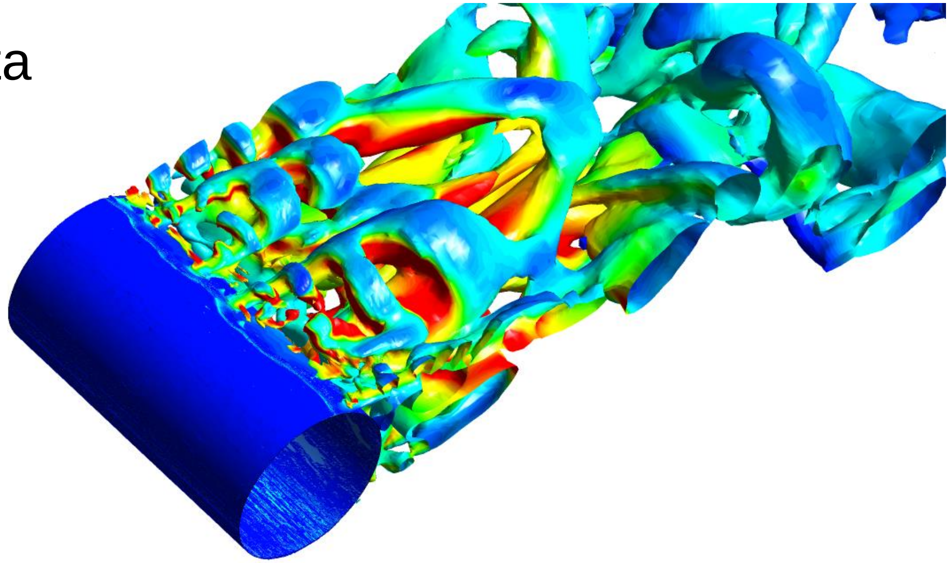
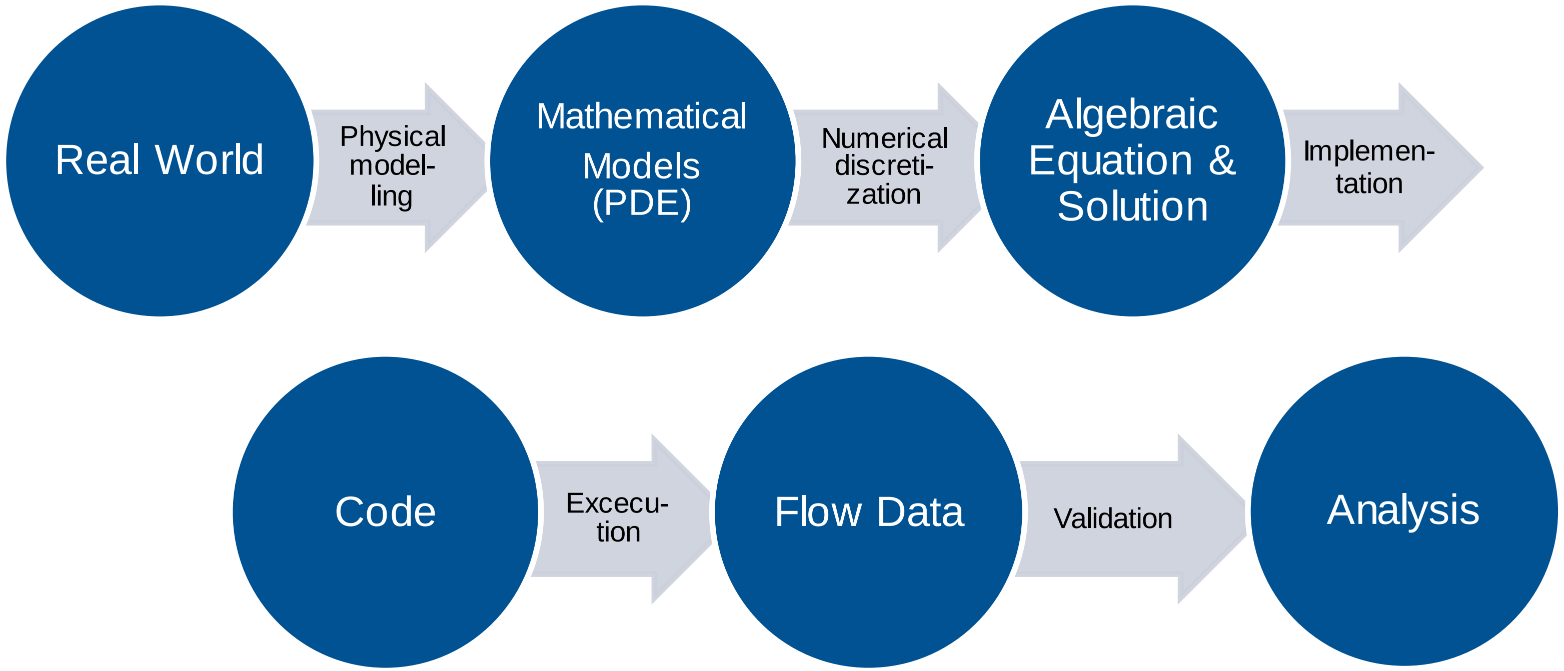


Physical Background

What is CFD?

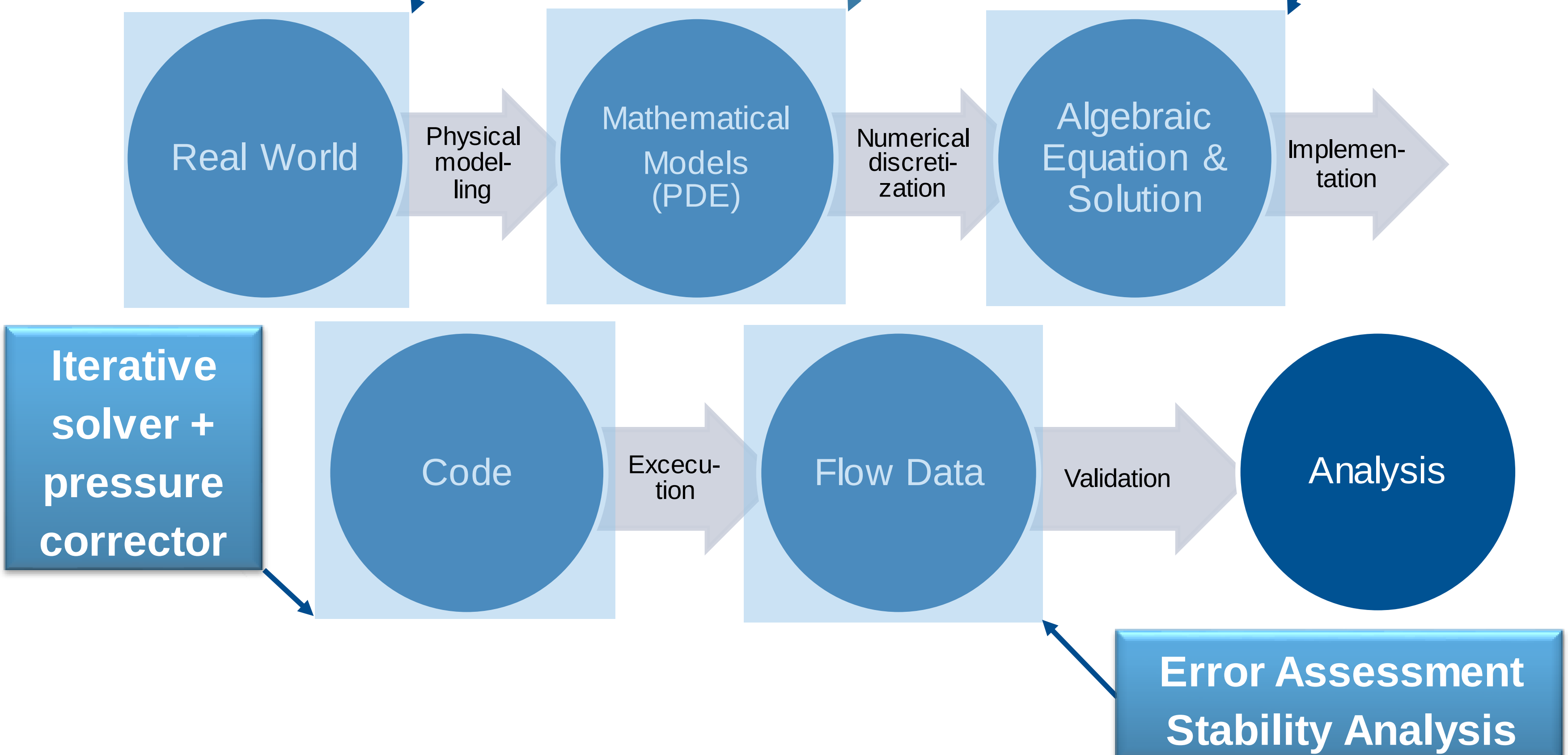
- Numerical simulation of fluid flow



Physical Backgrour

What is CFD?

- Selection of physical equations/models, discretization scheme



Modelling and Challenges

2D unsteady Navier – Stokes Equation for incompressible flow

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g_x,$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Temporal term: $\rho\frac{\partial u_i}{\partial t}$ → **Time Integration: Explicit Euler**

Convective term: $\rho u_j \frac{\partial u_i}{\partial x_j}$

~~Pressure term:~~ → **Not yet considered**

Viscous term: $\mu\frac{\partial^2 u_i}{\partial x_j^2}$

Convective term, Viscous term → **Spatial Discretization: Central Difference Scheme**

Modelling and Challenges

2D unsteady linear Advection - Diffusion Equation for incompressible flow

$$\frac{\partial u}{\partial t} = -U_0 \frac{\partial u}{\partial x} - V_0 \frac{\partial u}{\partial y} + \nu (\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2})$$

Time Integration (Explicit Euler):

$$u_i^{n+1} = u_i^n + dt * f_i^n$$

$$f_i^n = ad^n(i,j) + diff^n(i,j)$$

Spatial Discretization (Central Difference Scheme):

$$ad^n(i,j) = -U_0 \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} - \; -V_0 \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}$$

$$diff^n(i,j) = \nu \frac{u_{i+1,j} - u_{i,j} + u_{i-1,j}}{\Delta x^2} + \nu \frac{u_{i,j+1} - u_{i,j} + u_{i,j-1}}{\Delta y^2}$$

Advection - Diffusion Equation

Stability Analysis

- **EE** scheme in time and **CDS** in space for 1D Advection - diffusion equation

$$CFL = \frac{U\,\Delta t}{\Delta x}$$

$$D = \frac{\nu \Delta t}{\Delta x^2}$$

$$\frac{u_i^{n+1} - u_i^n}{2\Delta t} = -u_0 \frac{u_{i+1}^n - u_{i-1}^n}{2\,\Delta x} + \nu \frac{u_{i+1}^n - u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$u_i^{n+1} = -\frac{U_0\,\Delta t}{2\,\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{\nu\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2\,u_i^n + u_{i-1}^n\right)$$

$\frac{1}{2}\boldsymbol{CFL}$; $\boldsymbol{D}$

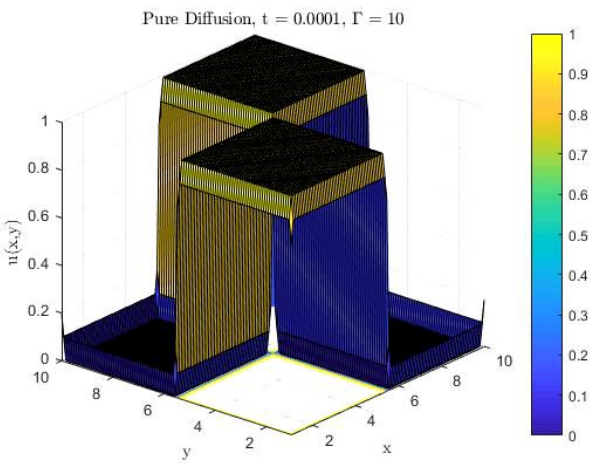
Fourier Transformation

$$G = \frac{\hat{u}^{n+1}}{\hat{u}^n} = 1 - 2D\left(1 - \cos(k\Delta x)\right) - i\;CFL\;sin(k\Delta x) < 1$$

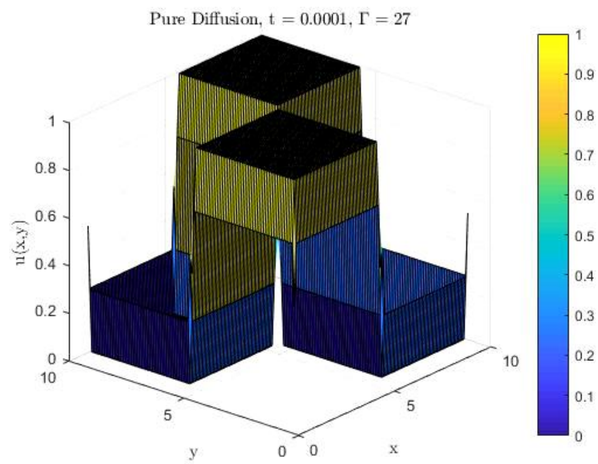
Advection - Diffusion Equation

Pure Diffusion $|G| = |1 - 2D\cos(k\Delta x)| < 1$ → Conditionally stable

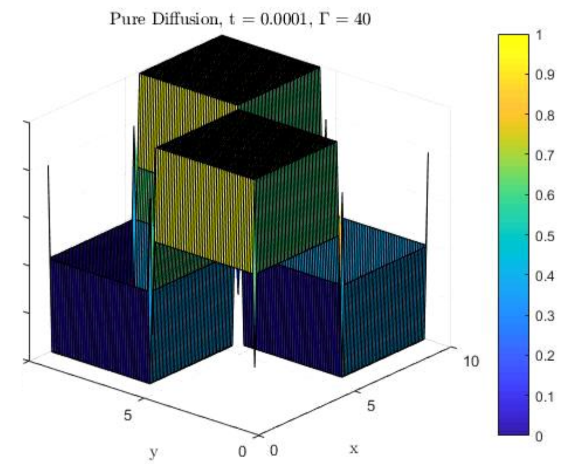
$D = 0.1$



$D = 0.27$



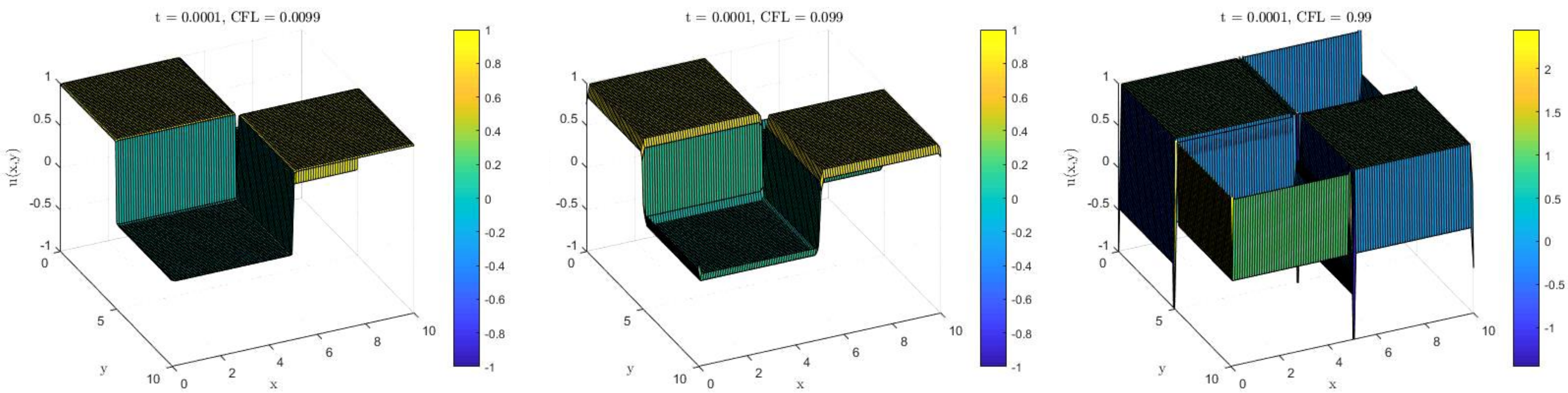
$D = 0.4$



Increasing Diffusion Number

Advection - Diffusion Equation

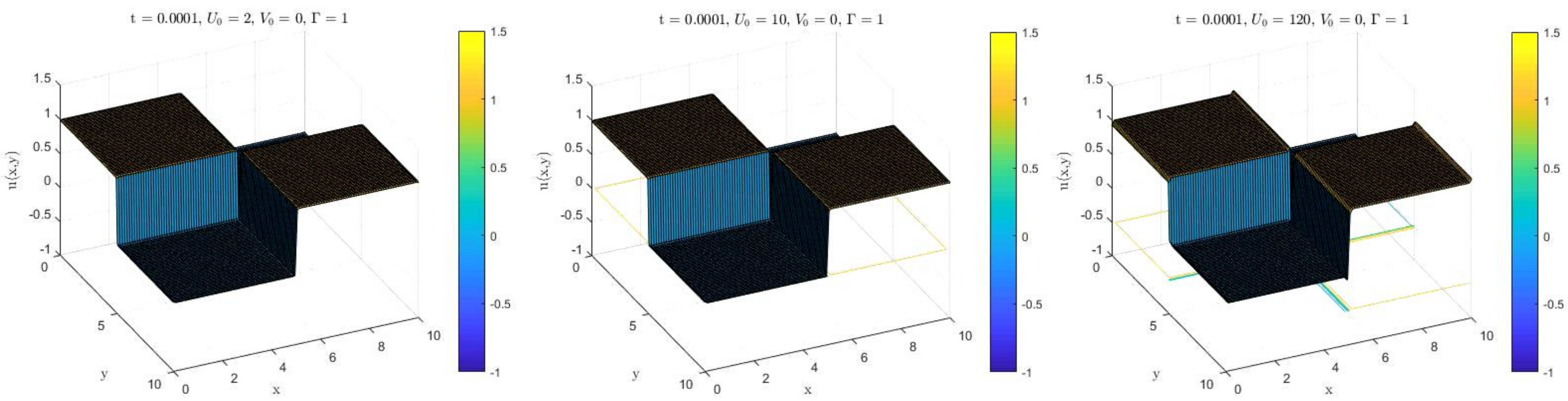
Pure Advection $G = 1 - i\,CFL\,sin(k\Delta x)$ → $|G| > 1$ → unstable



Increasing CFL number

Advection - Diffusion Equation

Dispersive Effect



Increasing Transport Velocity U_0

Advection - Diffusion Equation

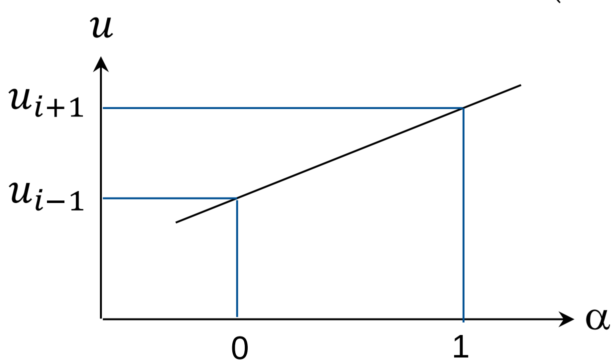
Advection – Cell Peclet Number $Pe|_{cell}$

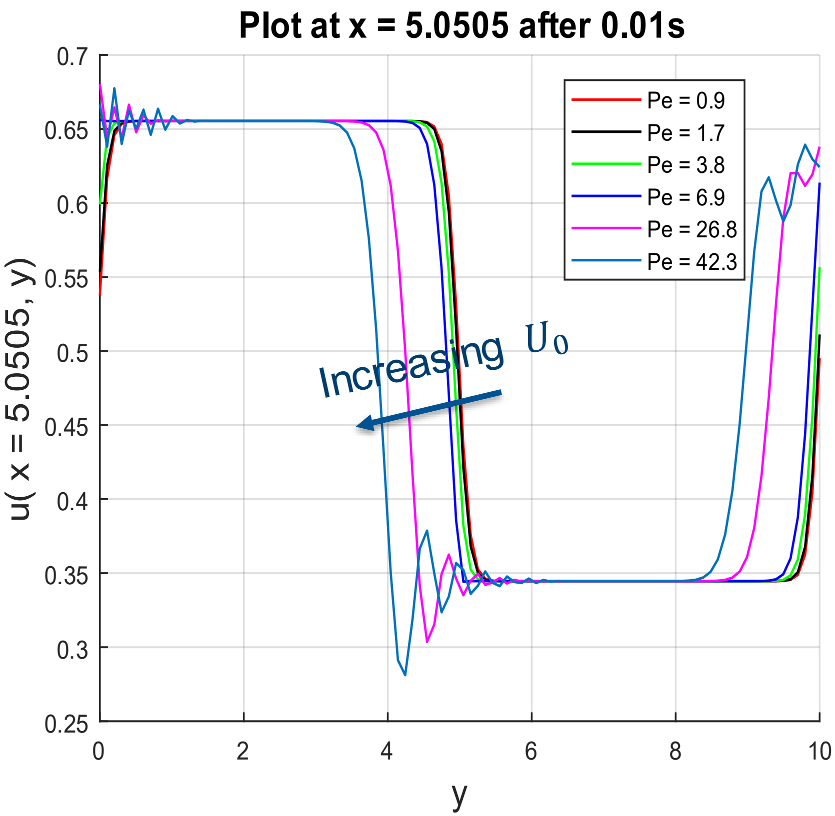
$$\frac{U_0}{2\Delta x}(u_{i+1} - u_{i-1}) = \frac{\nu}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1})$$

$$u_i = \frac{1}{2}[(1 - Pe|_{cell})u_{i+1} + (1 + Pe|_{cell})u_{i-1}]$$

$$Pe|_{cell} = \frac{U_0 \Delta x}{\nu}$$

$$u_i = a u_{i+1} + (1-\alpha)u_{i-1}, \qquad with\ \alpha = \frac{1}{2}\left(1 - \frac{Pe|_{cell}}{2}\right)$$





Two Challenges

2D unsteady Navier – Stokes Equation for incompressible flow

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g_x,$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Temporal term: $\rho\frac{\partial u_i}{\partial t}$ ← **1. Challenge Time integration**

Convective term: $\rho u_j\frac{\partial u_i}{\partial x_j}$

Pressure term: $-\frac{\partial p}{\partial x_i}$ ← **2. Challenge Pressure computation**

Viscous term: $\mu\frac{\partial^2 u_i}{\partial x_j^2}$

Time integration with Runge – Kutta method

2D Vectorial Nonlinear Convection-Diffusion Equation

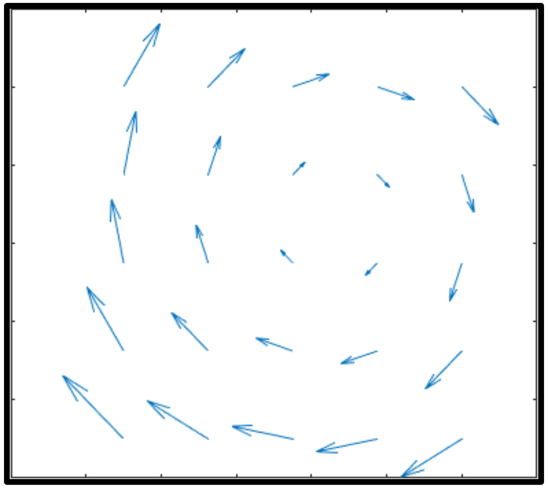
Governing Equations:

$$\frac{\partial u}{\partial t} = -u_{tr}\frac{\partial u}{\partial x} - v_{tr}\frac{\partial u}{\partial y} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} = -u_{tr}\frac{\partial v}{\partial x} - v_{tr}\frac{\partial v}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$u_{tr}(x, y) = f(y)$$

$$v_{tr}(x, y) = f(x)$$



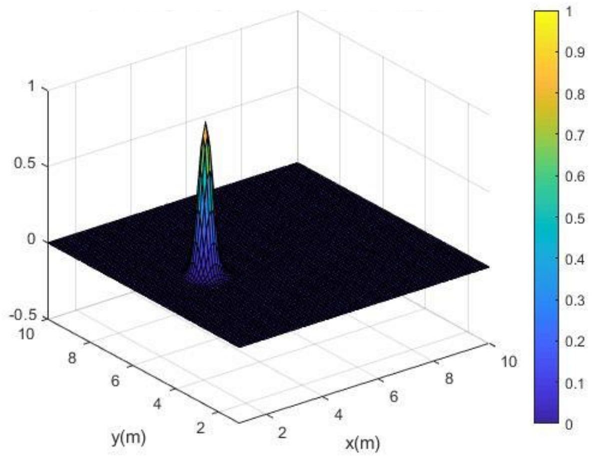
$$U_{tr} = (u_{tr}, v_{tr})$$

Domain: [0,Lx],[0,Ly],[0,T]

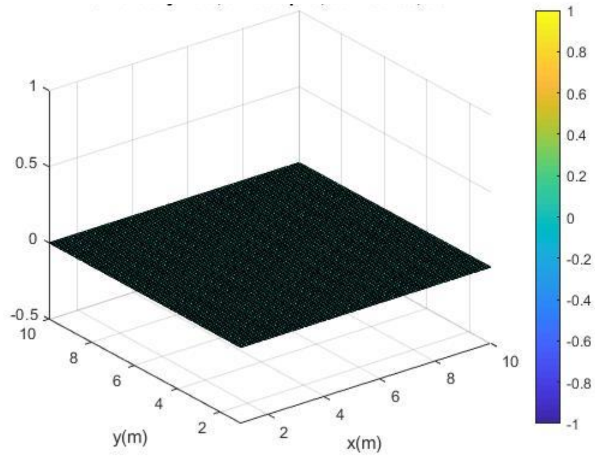
Boundary Conditions: Periodic boundary conditions

2D Vectorial Nonlinear Convection-Diffusion Equation

Initial condition:



U initial velocity field



V initial velocity field

2D Vectorial Nonlinear Convection-Diffusion Equation

Numerical Schemes:

Spatial Discritization : Central-Difference-Scheme

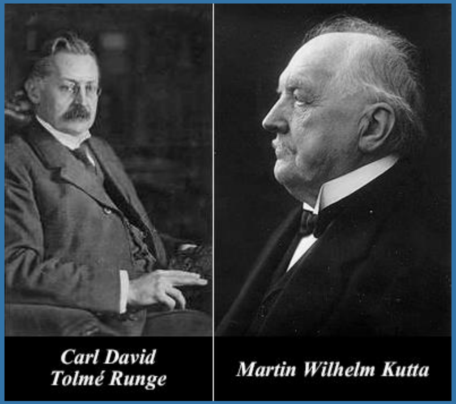
Time Integration : Runge Kutta 3 & Euler Explicit

Pros:

- Better accuracy.
- Possible larger time steps

Cons:

- More expensive
- Memory inefficient

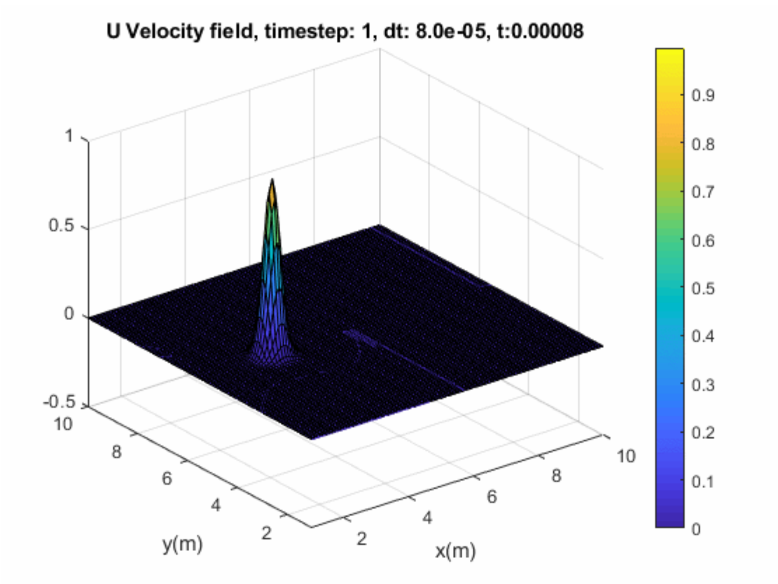


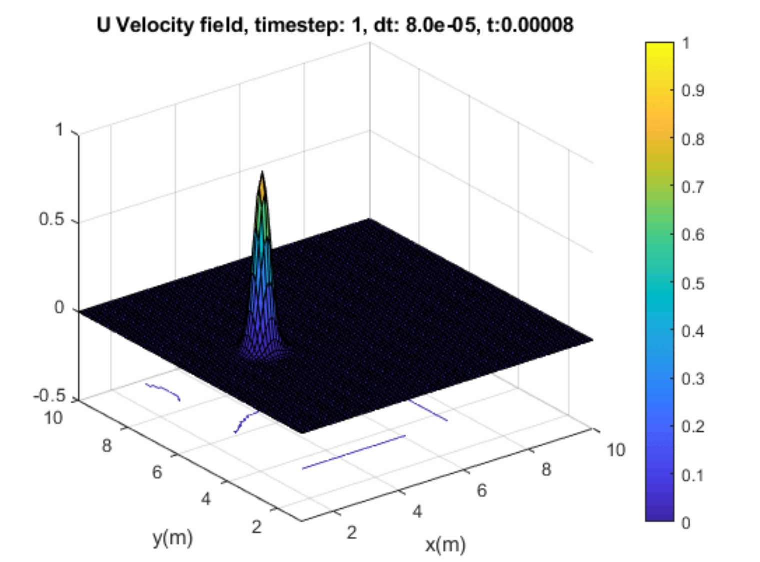
2D Vectorial Nonlinear Convection-Diffusion Equation

Case 1: Pure Convection

Euler..

RK3



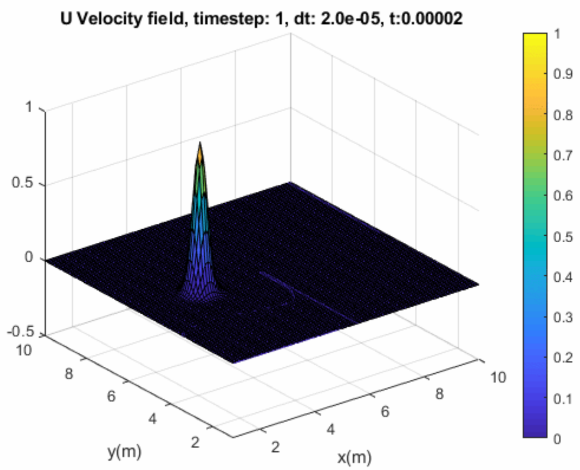


- **Wiggle occursion : Pe → ∞**
- **Decrease in U magnitude : Spatial discritization scheme**

2D Vectorial Nonlinear Convection-Diffusion Equation

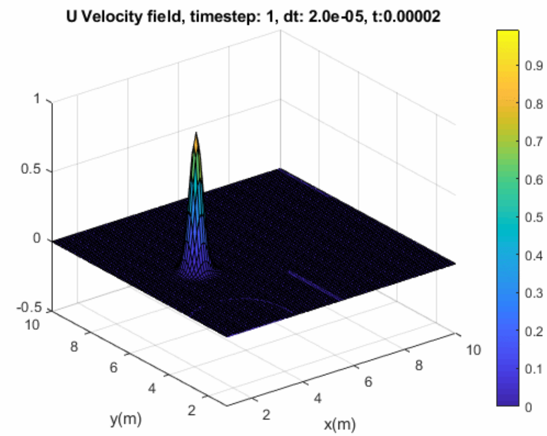
Case 2: Convection-Diffusion

CFL = 0.1



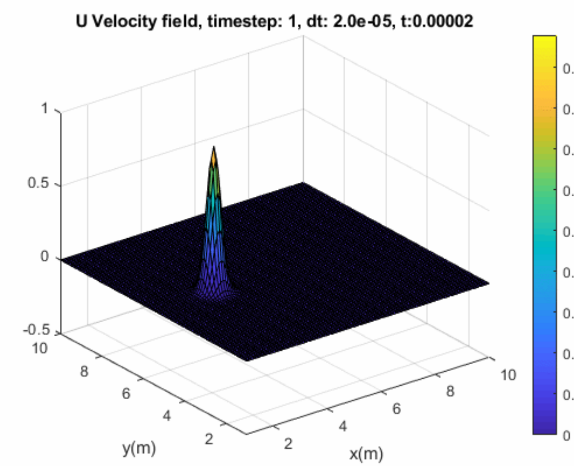
Viscosity = 0

Peclet = ∞



Viscosity = 10

Peclet = 0.5



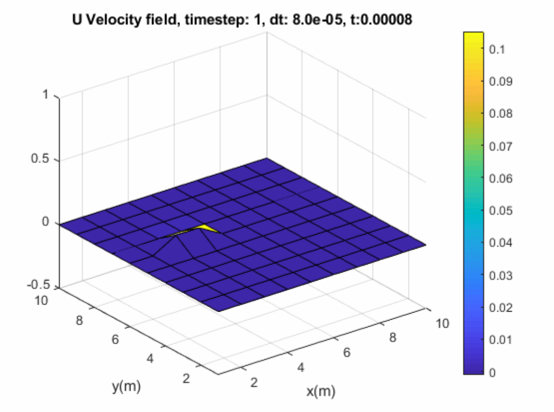
Viscosity = 30

Peclet = 0.166

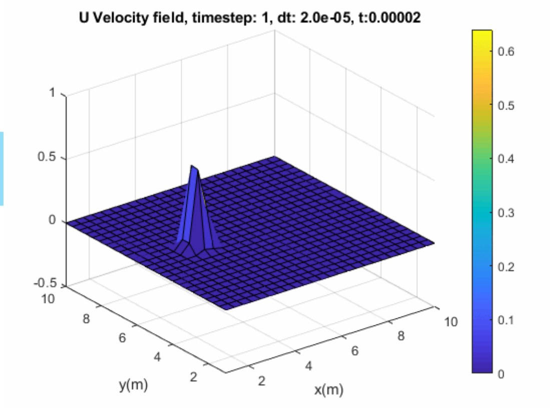
2D Vectorial Nonlinear Convection-Diffusion Equation

Case 3: Variation in Grid Number

CFL ⇓ ⇑ Accuracy

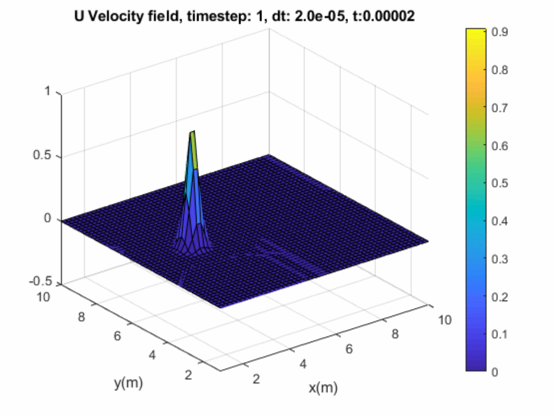


m = n = 10

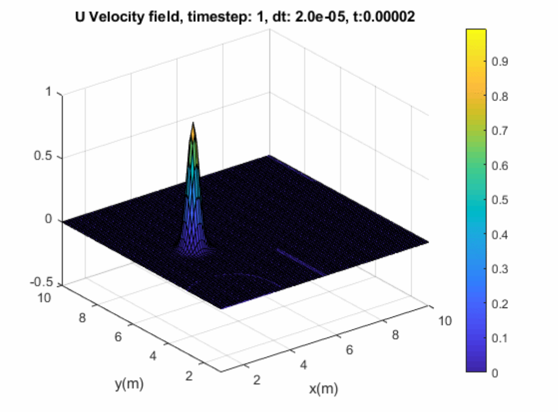


m = n = 25

m = n = 50



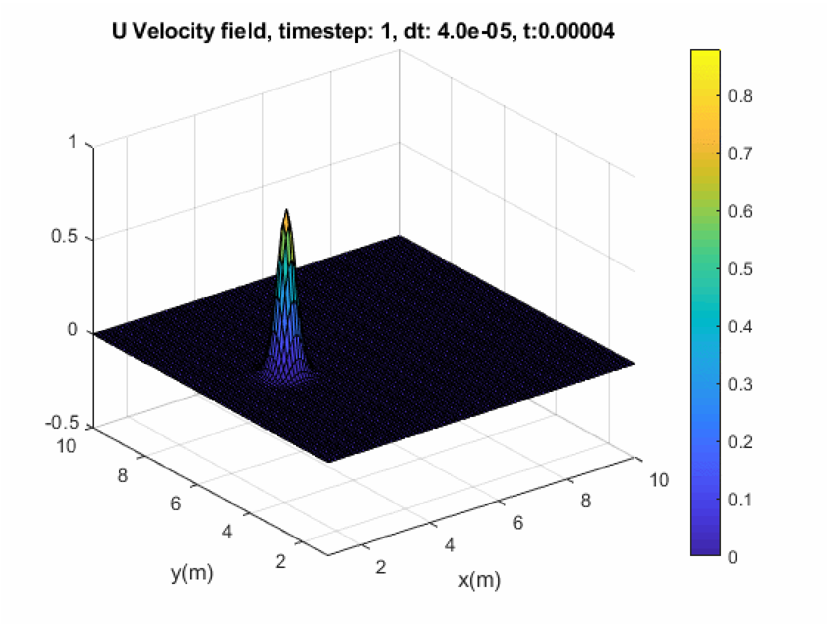
m = n = 100



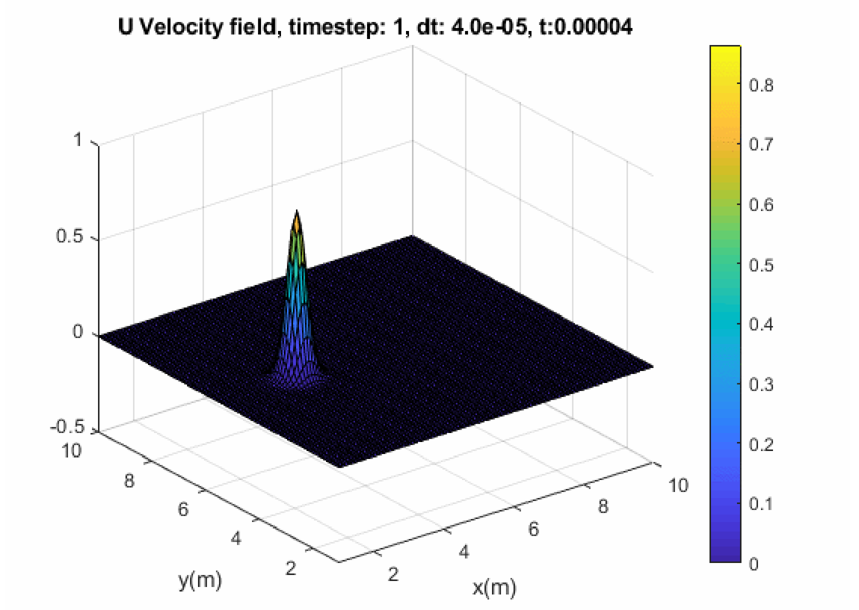
2D Vectorial Nonlinear Convection-Diffusion Equation

Case 4: Viscosity = 100

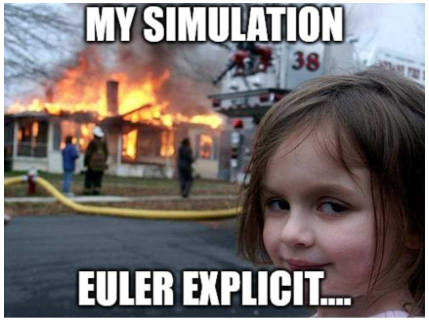
RK3



Euler

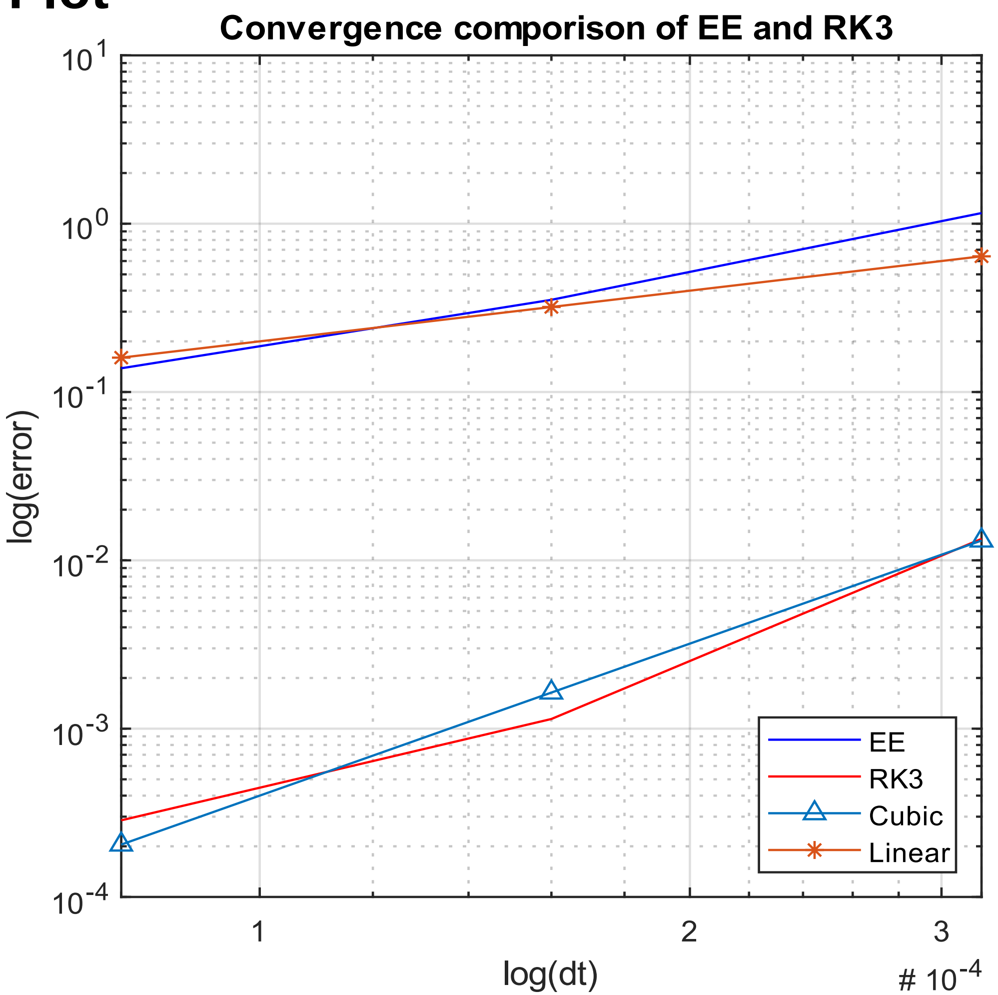


Stability problem occurs



Time integration with Runge – Kutta method

Convergence Plot



Pressure correction with continuity equation

- 2D unsteady Navier-Stokes equation is considered for uncompressible flow:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g_x,$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

- For the gravity driven channel problem, we can express the steady-state *u(y)* characteristic:

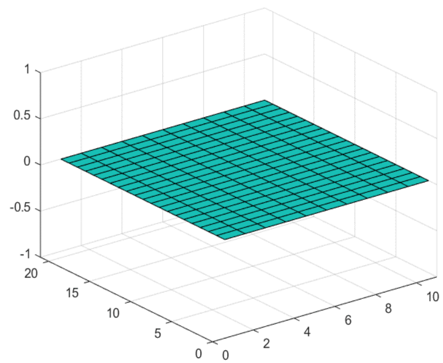
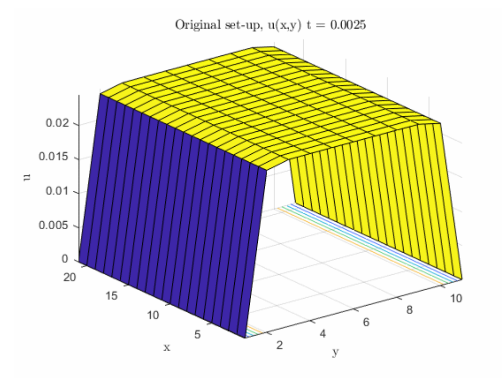
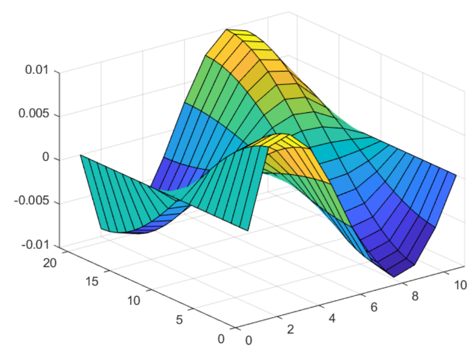
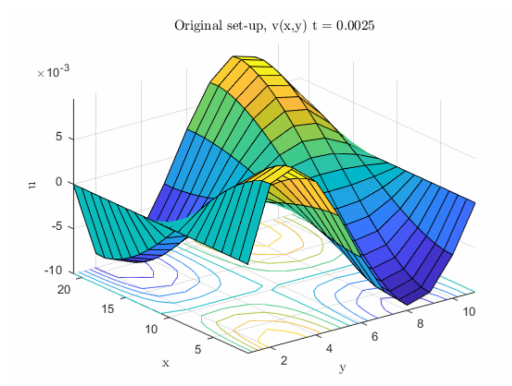
$$\underbrace{\frac{\partial u}{\partial t}}_{0} = -\frac{1}{\rho}\underbrace{\frac{\partial p}{\partial x}}_{0} - u\underbrace{\frac{\partial u}{\partial x}}_{0} - \underbrace{v}_{0}\frac{\partial u}{\partial y} + \nu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{0} + \frac{\partial^2 u}{\partial y^2}\right) + g_x$$

- with the original parameter set-up, peak velocity can be captured by the analytical solution for the **channel** problem:

$$u_{max} \approx 4{,}9$$

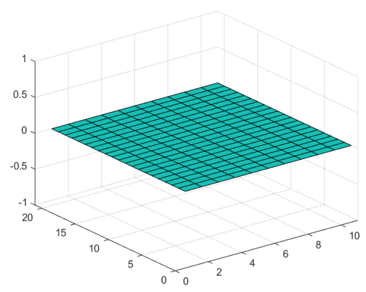
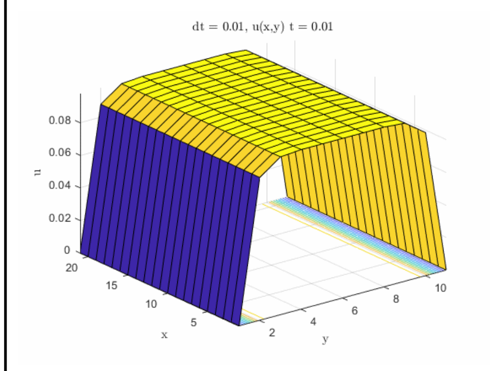
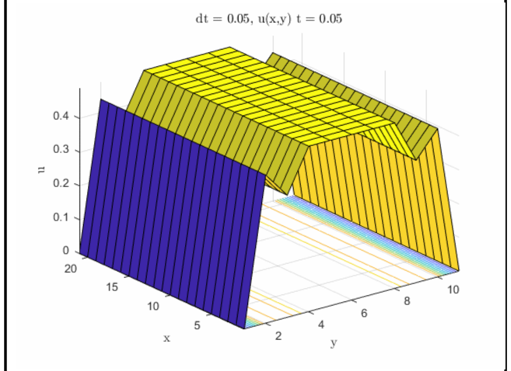
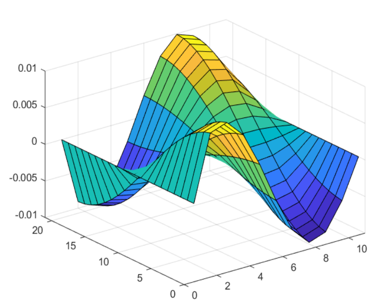
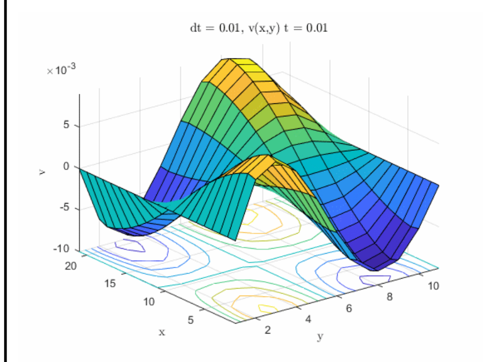
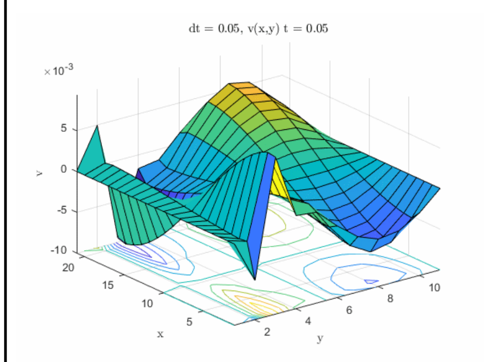
Original set-up for channel flow

dt	0.0025
dx	0.5
dy	0.2
ρ	1
υ	1
nrOfSt	2000

	Initial State	Velocity distributions
u(x,y)		
v(x,y)		

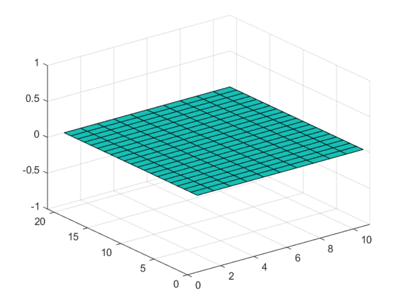
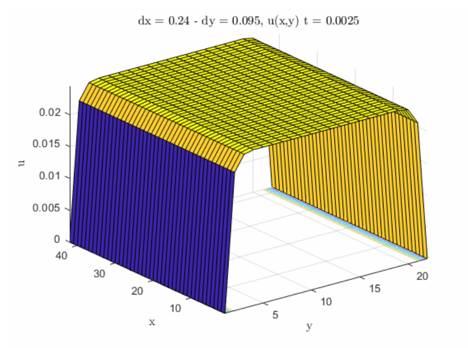
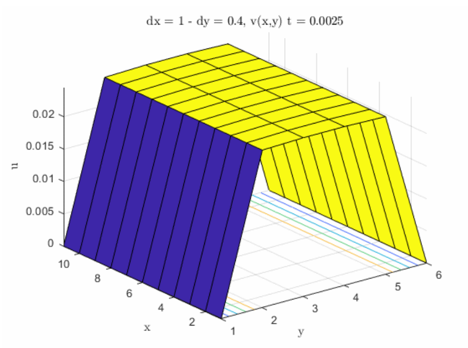
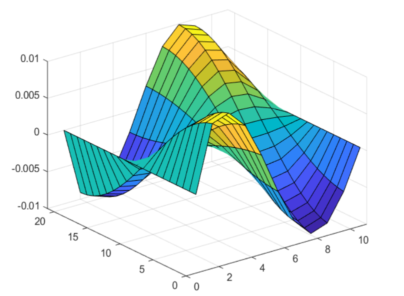
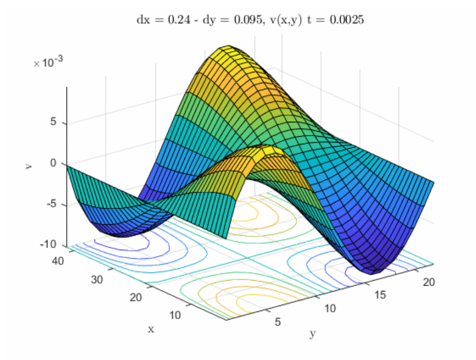
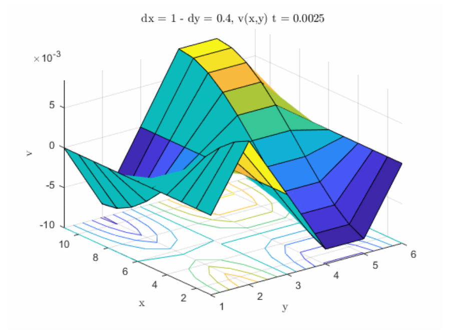
Time discretization for channel flow

dt	dt ↑
dx	0.5
dy	0.2
ρ	1
υ	1
nrOfSt	steps

	Initial State	dt = 0.01, steps = 300	dt = 0.05, steps = 30
u(x,y)			
v(x,y)			

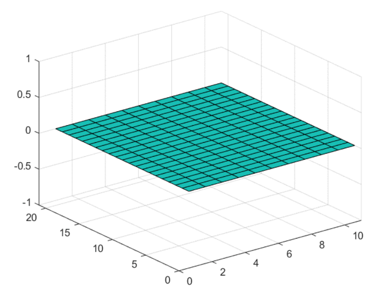
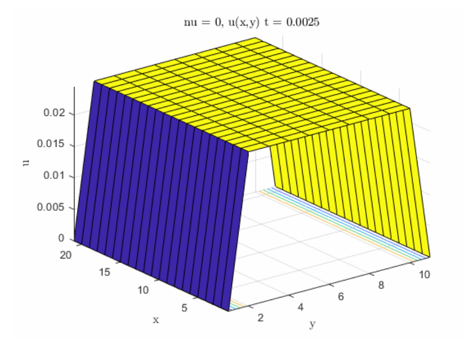
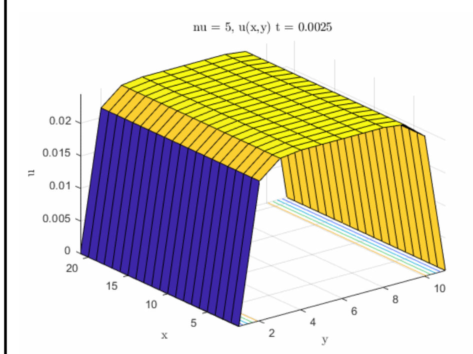
Spatial discretization for channel flow

dt	0.0025
dx	dx
dy	dy
ρ	1
υ	1
nrOfSt	500

	Initial State	dx = 0.24 – dy = 0.095	dx = 1 – dy = 0.4
u(x,y)			
v(x,y)			

Change of viscosity for channel flow

dt	0.0025
dx	0.5
dy	0.2
ρ	1
υ	υ
nrOfSt	1000

<table>
<tr><th></th><th>Initial State</th><th>$\upsilon = 0$</th><th>$\upsilon = 5$</th></tr>
<tr><td>u(x,y)</td><td></td><td>
</td><td>
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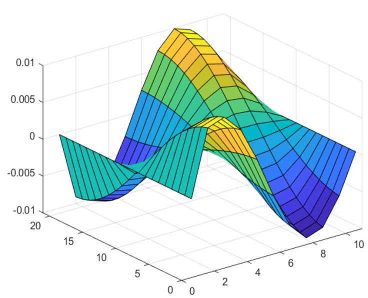
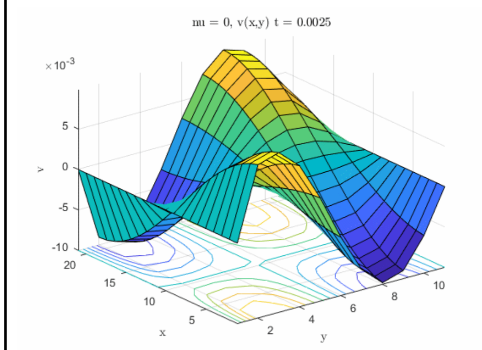
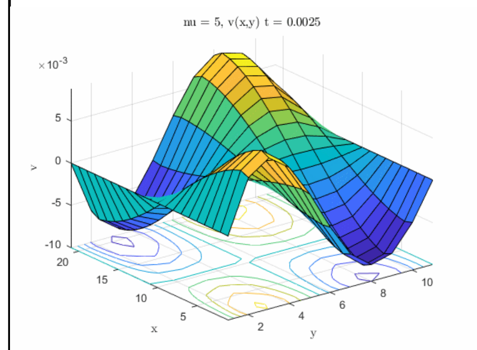
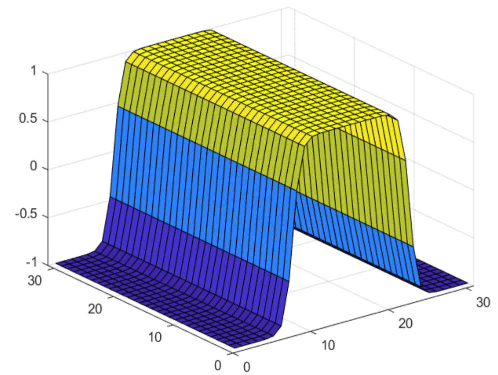
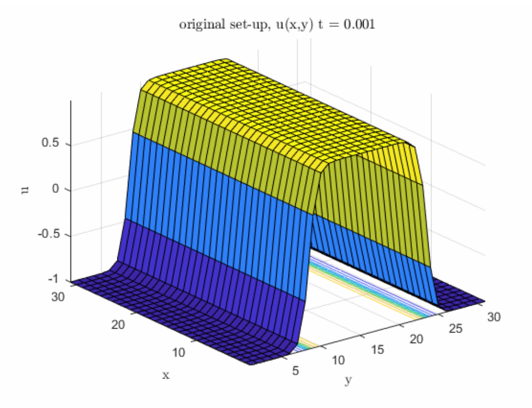
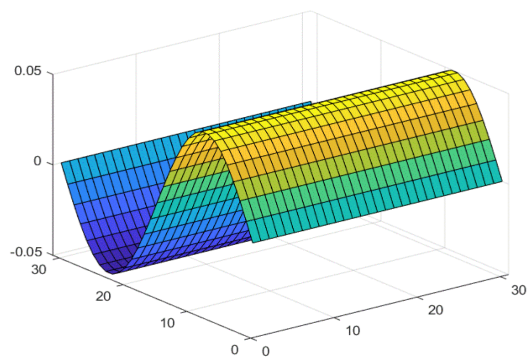
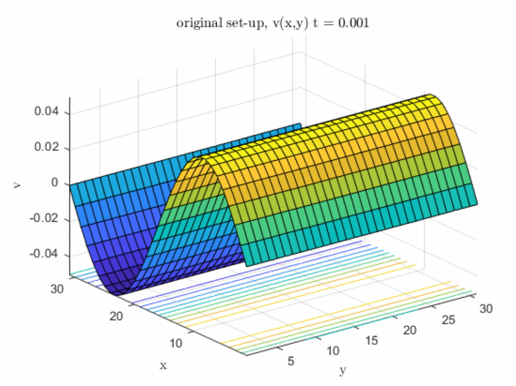
<tr><td>v(x,y)</td><td></td><td>

not damped this time!</td><td>
</td></tr>
</table>

with high enough viscosity values, scheme becomes unstable!

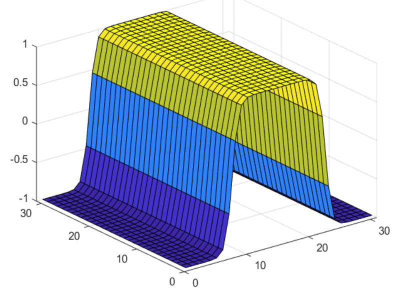
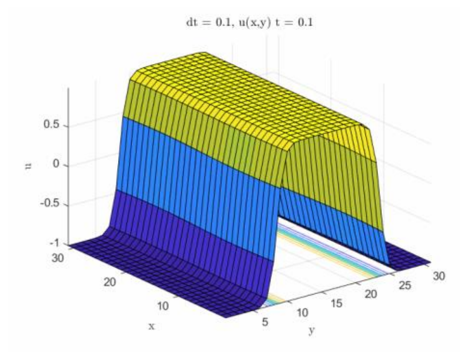
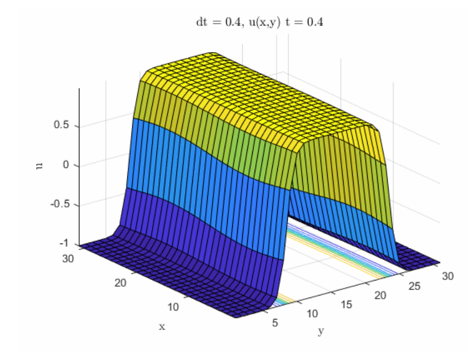
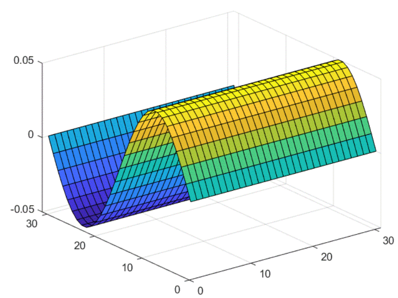
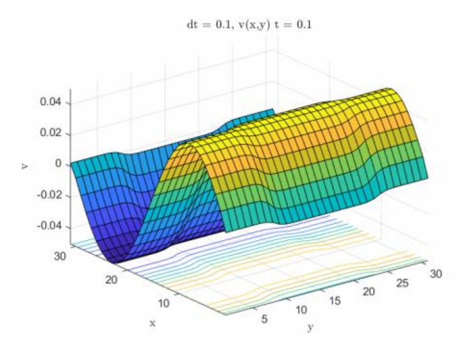
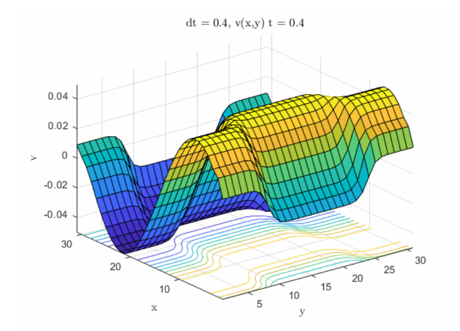
Original set-up for shear flow

dt	0.001
dx	0.209
dy	0.209
ρ	1
υ	0.01
nrOfSt	2000

	Initial State	Velocity distributions
u(x,y)		
v(x,y)		

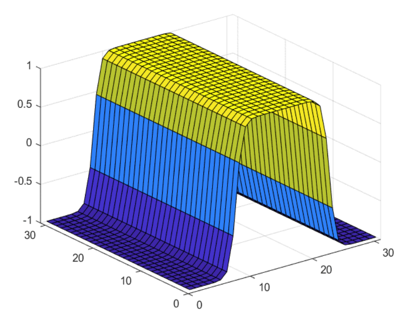
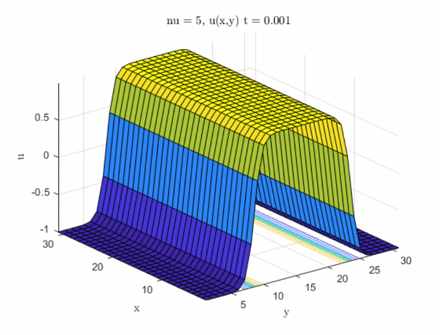
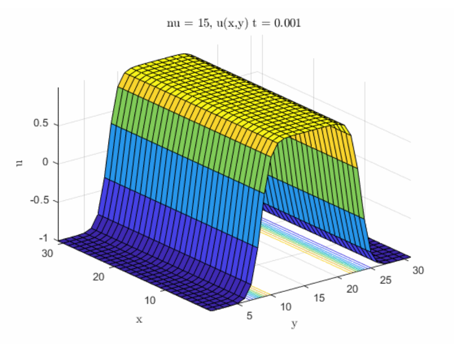
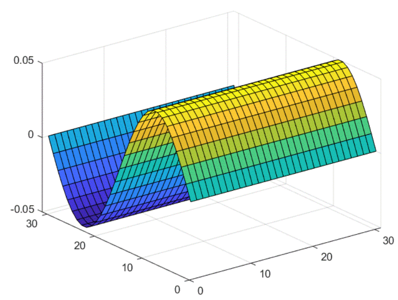
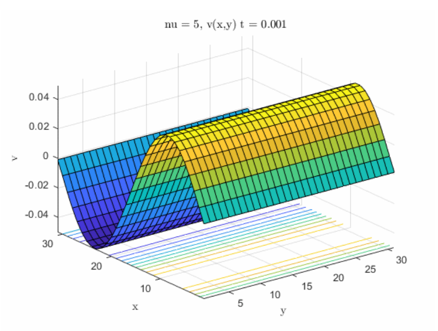
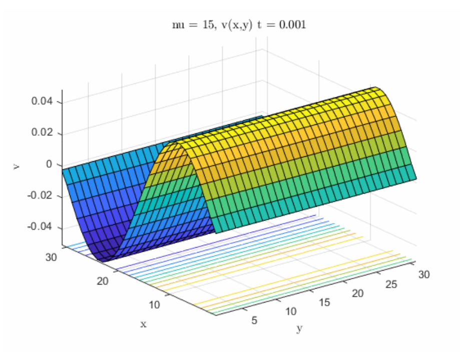
Time discretization for shear flow

dt	dt
dx	0.209
dy	0.209
ρ	1
υ	0.01
nrOfSt	steps

	Initial State	dt = 0.1, steps = 2000	dt = 0.4, steps = 50
u(x,y)			
v(x,y)			

Change of viscosity for shear flow

dt	0.001
dx	0.209
dy	0.209
ρ	1
υ	υ
nrOfSt	1000

	Initial State	υ = 5	υ = 15
$u(x,y)$			
$v(x,y)$			

with high enough viscosity values, scheme becomes unstable!

Pressure correction with continuity equation

Summary

Time discretization

- Large effect to stability properties
- Choosing too large step size leads to instability
- Manipulated step size might have an impact of computed fields characteristic

Spatial discretization

- No effect for stability or field variable distribution in our tested configurations

Material parameters

- Viscosity affects the development of velocity and pressure distribution
- Large damping effect with increased viscosity
- Choosing too large viscosity leads to instability
- Density does not apply an effect in our configurations

Two Improvements

2D unsteady Navier – Stokes Equation for incompressible flow

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g_x,$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} =$$

Temporal term: $\rho\frac{\partial u_i}{\partial t}$

Convective term: $\rho u_j\frac{\partial u_i}{\partial x_j}$

Pressure term: $-\frac{\partial p}{\partial x_i}$

Viscous term: $\mu\frac{\partial^2 u_i}{\partial x_j^2}$

2. Improvement
- **In sense of numerical efficiency**
- **Iterative Solver (Gauss Seidel) instead of direct solver**

1. Improvement
- **In sense of numerical accuracy**
- **Pressure correction**

Iterative Solver (Gauß – Seidel)

- Using previously computed values in the solution vector of the same iteration
- Computing complexity = n^2
- Accuracy: approximated value

$$x_i^{(k+1)} = -\frac{1}{a_{ii}}\left(\sum_{j=1}^{i-1} a_{ij}\, x_j^{(k+1)} + \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k)} - b_i\right)$$

$$x_1^{(k+1)} = -\frac{1}{a_{11}}\left(\ldots + a_{12}\, x_2^k + a_{13}\, x_3^k + a_{14}\, x_4^k - b_1\right)$$

$$x_2^{(k+1)} = -\frac{1}{a_{33}}\left(a_{21}\, x_1^{k+1} + \cdots + a_{23}\, x_3^k + a_{24}\, x_4^k - b_2\right)$$

$$x_3^{(k+1)} = -\frac{1}{a_{33}}\left(a_{31}\, x_1^{k+1} + a_{32}\, x_2^k + \cdots + a_{34}\, x_4^k - b_3\right)$$

Direct Solver (Gauß – Jordan)

- Is a modification of the Gauss – Elimination Method
- Computing complexity = $\frac{n^3}{3} + n^2$
- Accuracy: exact solution

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{Reduced\ Row\ Echolon\ Form} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Iterative Solver (Gauß – Seidel)

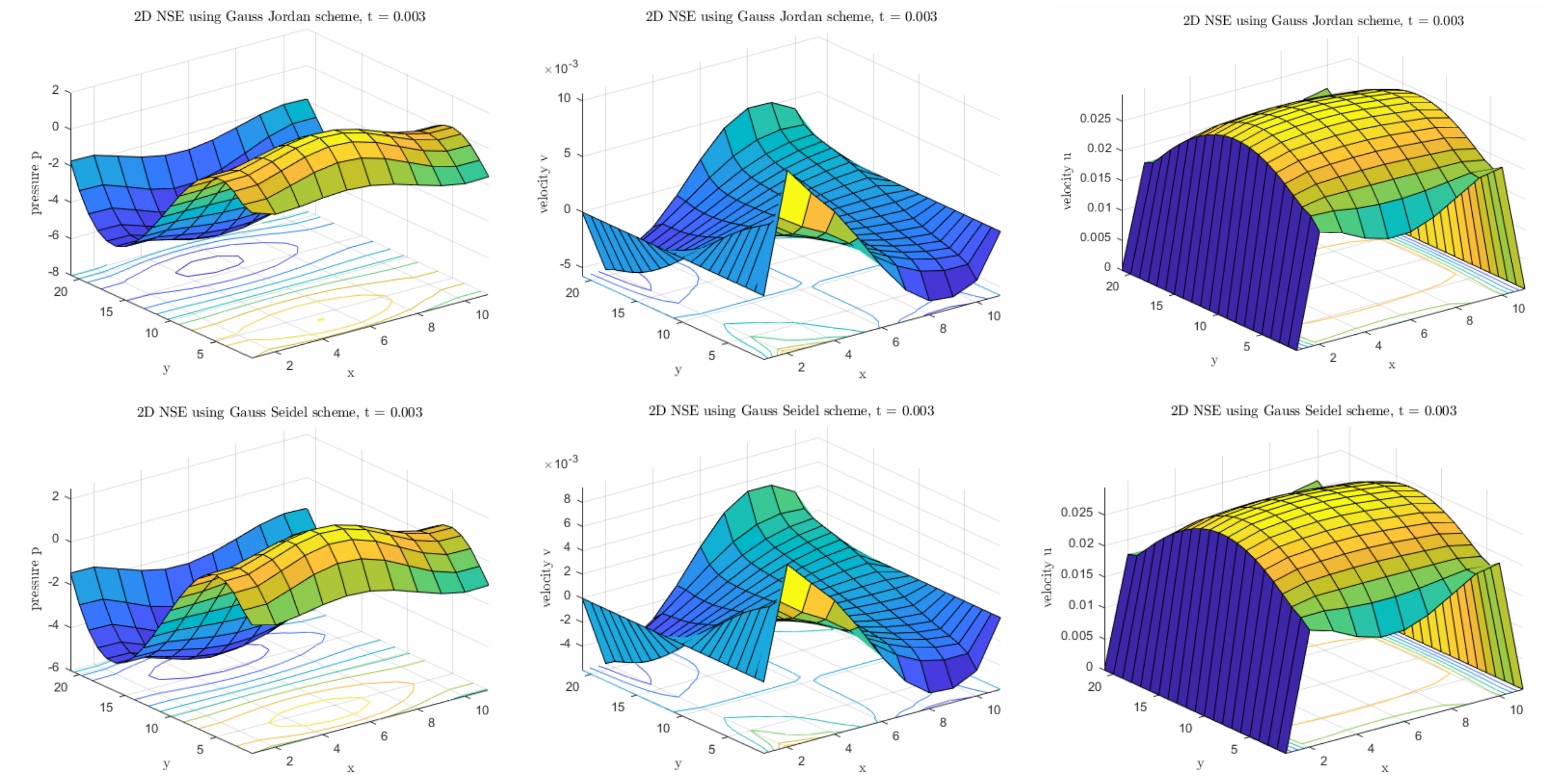
Default Parameters

- Channel Flow:
  - dt = 0.0025
  - Time steps = 600
  - m = 21
  - n = 11
  - Tolerance = 1e-4

- Shear Flow:
  - dt = 1e-3
  - Time steps = 600
  - m = 31
  - n = 31
  - Tolerance = 1e-4

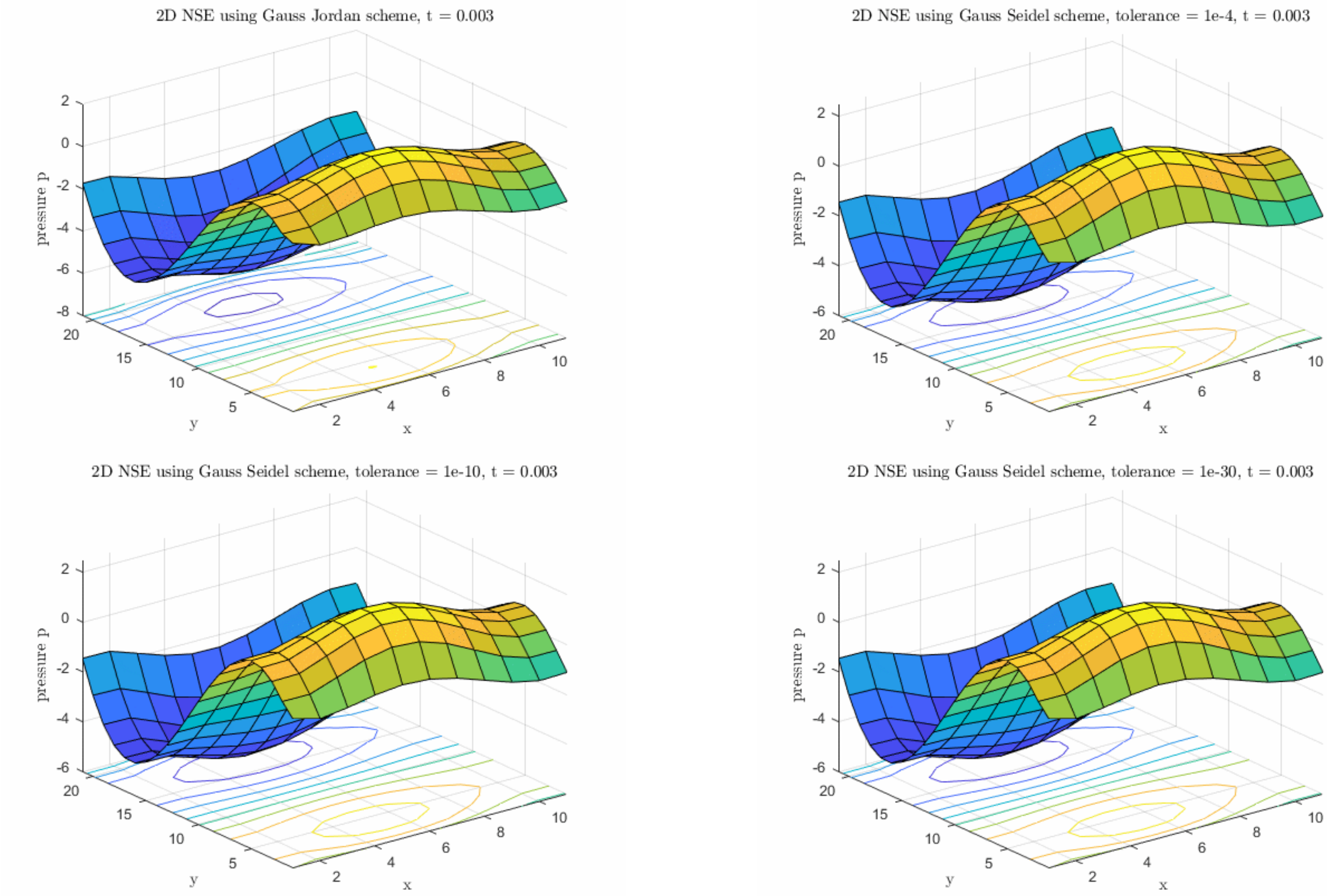
Iterative Solver (Gauß – Seidel)

Validation



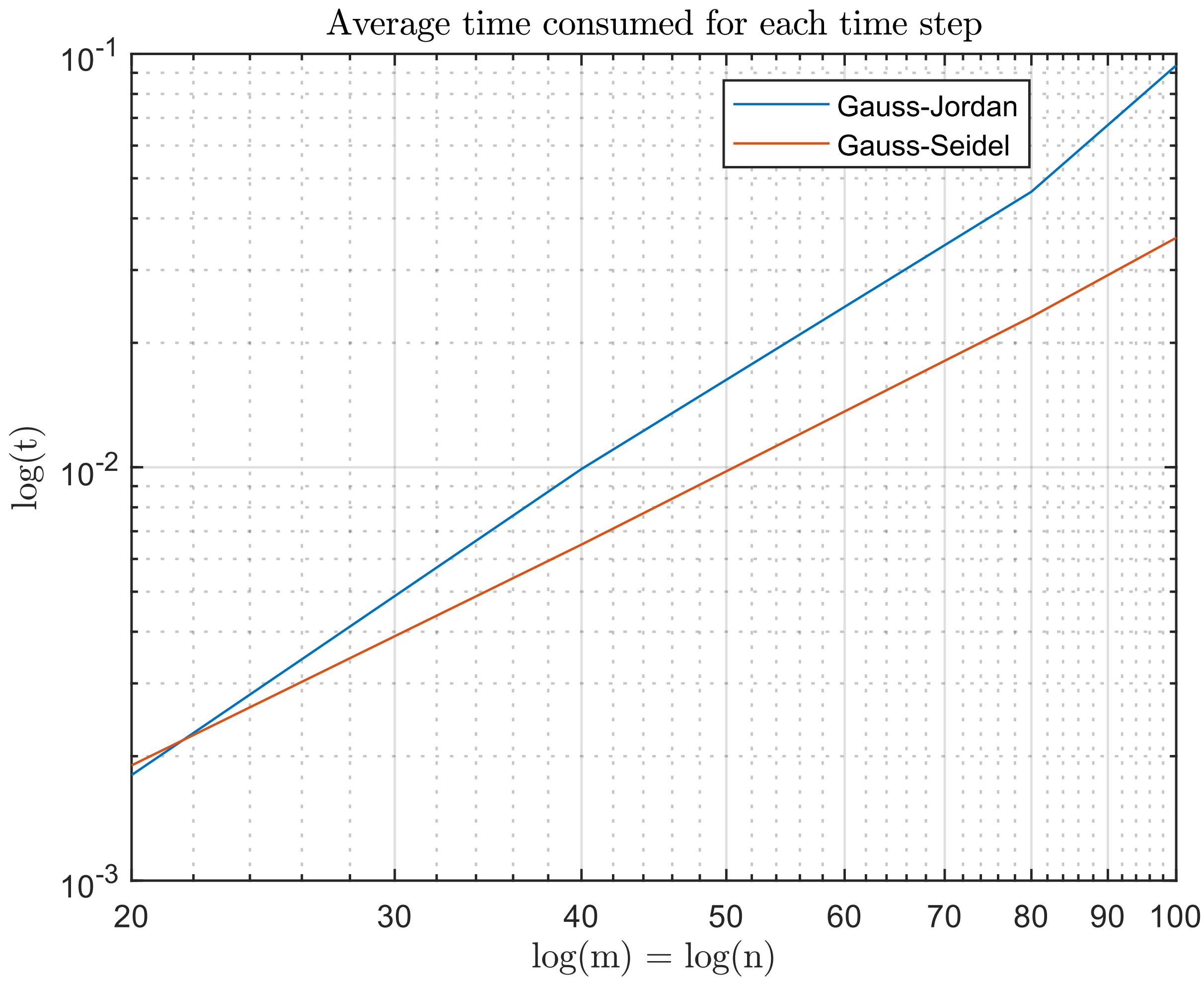
Iterative Solver (Gauß – Seidel)

Decreasing Tolerance



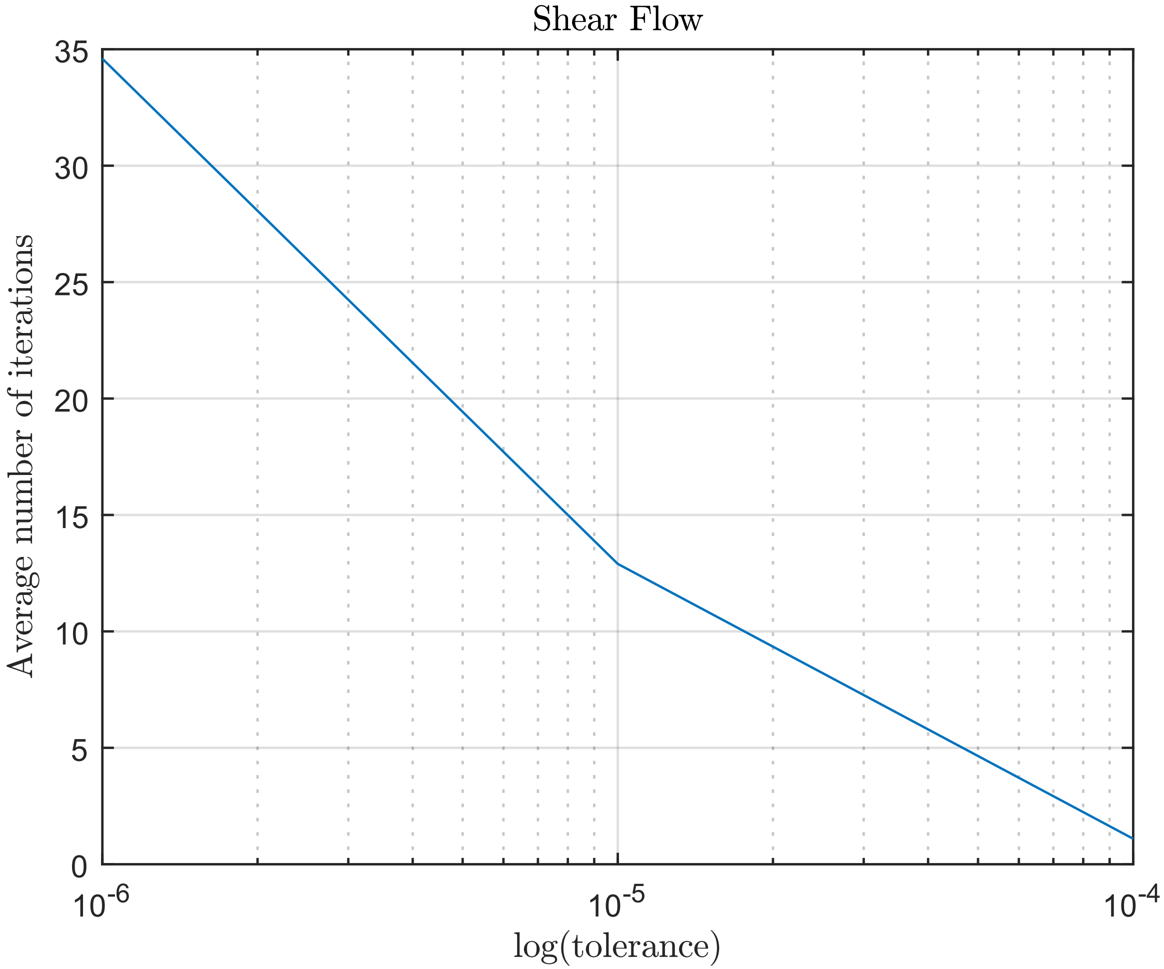
Iterative Solver (Gauß – Seidel)

Computational Complexity



Iterative Solver (Gauß – Seidel)

Average Number of GS Iterations per Time Step

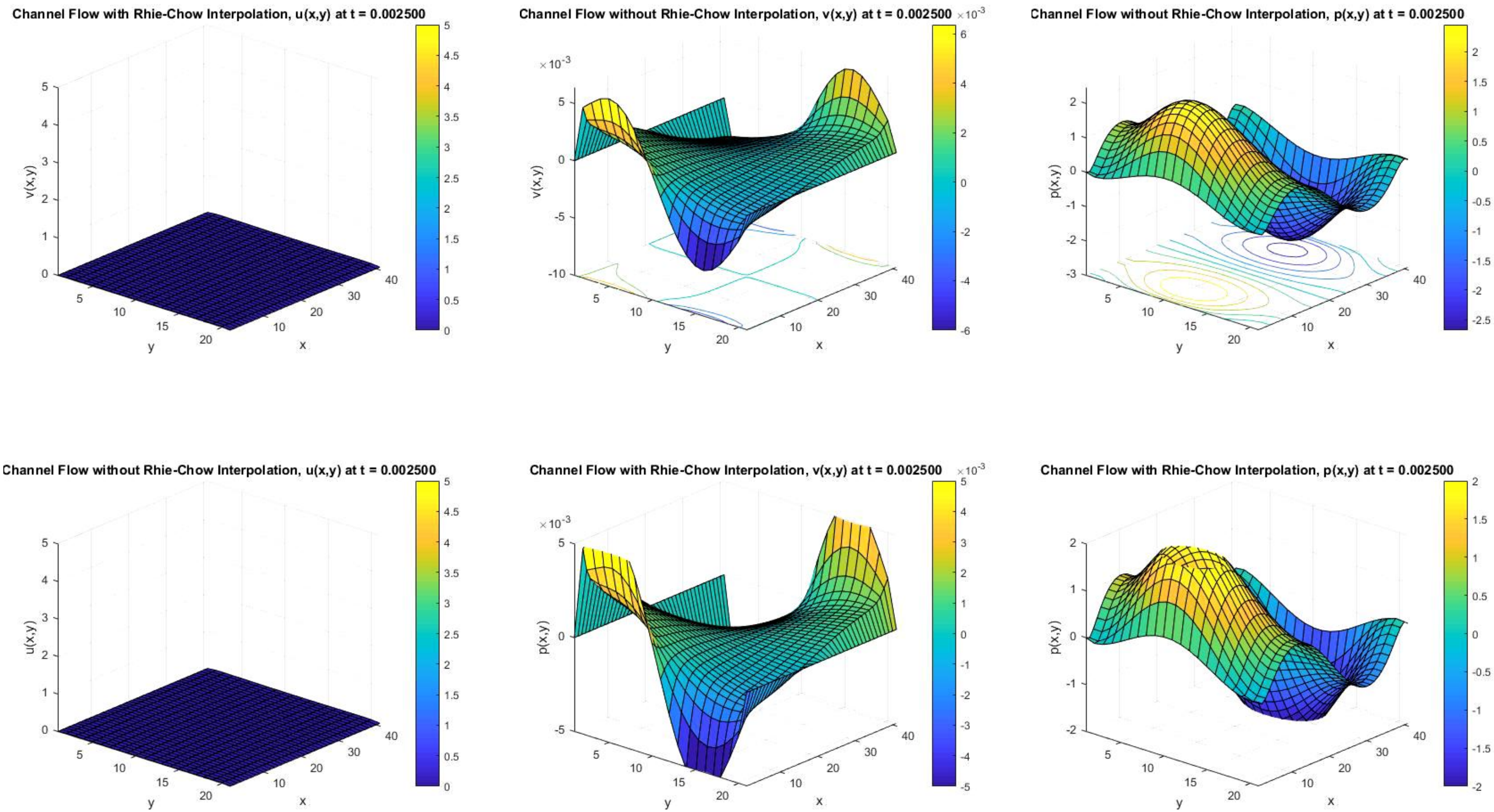


Iterative Solver (Gauß – Seidel)

Summary

- Gauss Seidel is an iterative scheme that give an approximate solution
- Decreasing the tolerance will give a result almost exact to Gauss Jordan
- Computational complexity of Gauss Jordan is higher than Gauss Seidel with higher dimensions
- Average number of Gauss Seidel iterations per time step decreases with increasing tolerance

Rhie – Chow Interpolation



THANK YOU FOR YOUR ATTENTIONS

Group 9

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CFD Lab SS18

Munich, June 30, 2020